SEMESTER : **I**
ALLIED COURSE : **I - Mathematics**

Inst Hour	**: 5**
Credit	**: 3**
Code	**: 18K1MAM1**

NUMERICAL METHODS - I

(For B.Sc., Mathematics Major)

UNIT I:

The Solutions of Numerical Algebraic and Transcendental Equations – Bisection Method –The Iteration Method – The Method of False Position - Newton Raphson Method .
(Chapter 3-3.1-3.4)

UNIT II:

Solution of Simultaneous Linear Algebraic Equations: Introduction – Gauss Elimination Method – Gauss Jordan Method - Iterative Methods- Jacobi Method - Gauss Seidal Method of Iteration.
(Chapter 4- 4.1, 4.2, 4.2.1, 4.7- 4.9)

UNIT III:

Difference Equations: Definition, Order and Degree of Difference Equations – To find Complementary Functions and Particular Integrals of the type (i)a^x (ii) x^m (iii) $x^m a^x$ (Simple Problems)
(Chapter 10-10.1-10.6)

UNIT IV:

Numerical Solution of Ordinary Differential Equations – Solution by Taylor's Series – Picard's Method of Successive Approximations – Euler's Method and Modified Euler's Method – Second and Fourth order Runge - Kutta Method for First Order Ordinary Differential Equations.
(Chapter 11: Sections 11.5, 11.8, 11.9, 11.11, 11.12, 11.13)

UNIT V:

Numerical Solution of Partial Differential Equations – Elliptic Equations - Laplace's Equations – Jacobi's Method – Gauss Seidal Method – Parabolic Equations – Crank Nicholson Difference Method.
(Chapter 12: Sections -12.5, 12.6, 12.8, 12.9)

(In all the Units SIMPLE PROBLEMS ONLY)

Text Books:

1. Kandasamy.P, Thilagavathy, K,Gunavathi.K., Numerical Methods ,S.Chand & Company Ltd 2015.

Reference Books

[1] S.S.Sastry, Introductory Methods of Numerical Analysis,Prentice Hall of India Private Limited, Fourth Edition.

[2] M.K.Venkataraman, Numerical Analysis, The National Publishing Company, Madras, Fifth Edition.

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.

Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.

Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

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NUMERICAL METHODS - I

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UNIT-I

The solution of Numerical Algebraic and Transcendental equations

3.1 In the field of science and Engineering, the solution of equations of the form $f(x)=0$ occurs in many application. If $f(x)$ is a polynomial of degree two or three or four, exact formulae are available. But, if $f(x)$ is a transcendental function like $a+be^x+c\sin x+d\log x$ etc., the solution is not exact and we do not have formulae to get the solutions. When the coefficients are numerical values, we can adopt various numerical approximate methods to solve such algebraic and transcendental equations. We will see below some methods of solv such numerical equations. From the theory of equatio we recall to our memory the following theorem

If $f(x)$ is continuous in the interval (a,b) an if $f(a)$ and $f(b)$ are of opposite signs, then the equation $f(x)=0$ will have atleast one real root between a and

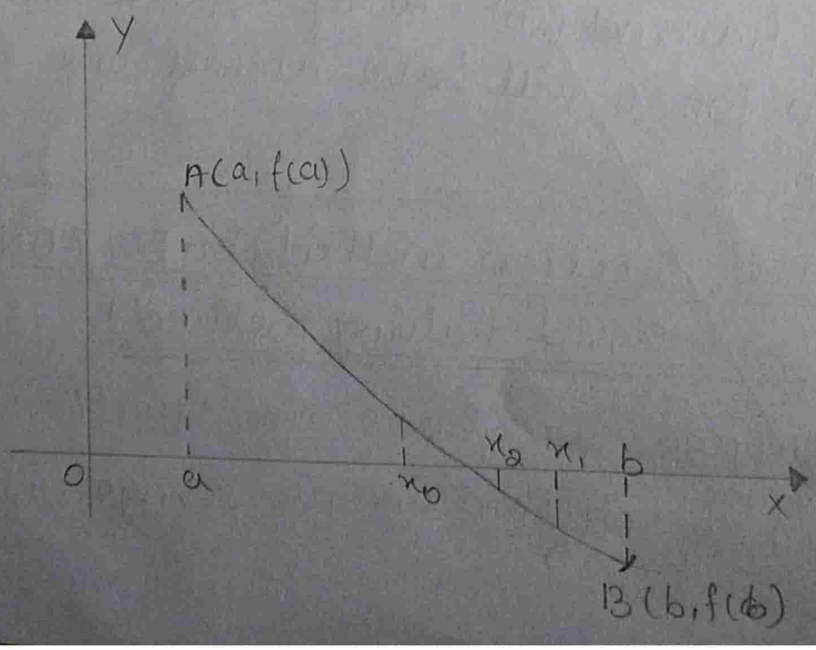
3.1.1. The Bisection method (or BOLZANO's method) (or Interval halving method)

AIM:

Suppose we have an equation of the form $f(x)=0$ whose solution in the range (a,b) is to be searched.

We also assume that $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are of opposite signs. atleast one real root between a and b should exist. For convenience, let $f(a)$ be positive and $f(b)$ be negative. Then atleast one root exists between a and b. As a first approximation, we assume that root to be $x_0 = \frac{a+b}{2}$ (mid point of the ends of the range) Now, find the sign of $f(x_0)$. If $f(x_0)$ is negative, the root lies between a and x_0. If $f(x_0)$ is positive, the root lies between x_0 and b. Any one of this is true. Suppose $f(x_0)$ is positive as shown in the Fig 3.1. then the root lies between x_0 and b and take the root as $x_1 = \frac{x_0+b}{2}$. Now $f(x_1)$ is negative. Hence the root lies between x_0 and x_1 and let the root be (approximate) $x_2 = \frac{x_0+x_1}{2}$. Now $f(x_2)$ is negative as in the Fig 3.1, then the root lies between x_0 and x_2 and $x_3 = \frac{x_0+x_2}{2}$ and so on

In this way taking the mid-point of the range as the approximate root, we form a sequence of approximate roots $x_0, x_1, x_2 \ldots$ whose limit of convergence is the exact root. However, depending on the precision required, we stop the process after some steps. Though simple, the convergence of this method is slow but s



Ex:1 Find the positive root of $x^3-x=1$ correct to four decimal places by bisection method.

Solution:

Let $f(x) = x^3 - x - 1$

$f(0) = (0)^3 - 0 - 1 = -1$ (-ve)

$f(1) = (1)^3 - 1 - 1 = -1$ (-ve)

$f(2) = (2)^3 - 2 - 1 = 5$ (+ve)

Hence a root lies between 1 and 2. We can take the range as (1, 2) and proceed. We can still shorten the range

$f(1.5) = 0.8750 = +ve$

$f(1) = -1 = -ve$

Hence, the root lies between 1 and 1.5

Let $x_0 = \frac{1+1.5}{2} = 1.2500$

$f(x_0) = f(1.25) = 0.29688$

Hence the root lies between 1.25 and 1.5

Now, $x_1 = \frac{1.25+1.5}{2} = 1.3750$

$f(x_1) = f(1.3750) = 0.22461 = +ve$

The root lies between 1.2500 and 1.3750

Now, $x_2 = \frac{1.2500+1.3750}{2} = 1.3125$

$f(x_2) = f(1.3125) = -0.051514$

Therefore, root lies between 1.3750 and 1.3125

Now $x_3 = \frac{1.3125+1.3750}{2} = 1.3438$

$f(x_3) = f(1.3438) = 0.082832 = +ve$

The root lies between 1.3125 and 1.3438

$x_4 = \frac{1.3125+1.3438}{2} = 1.3282$

$f(x_4) = f(1.3282) = 0.014898$

Therefore the root lies between 1.3125 and 1.3282

$$x_5 = \frac{1.3125 + 1.3282}{2} = 1.3204$$

$f(x_5) = f(1.3204) = -0.0183.40$

The root lies between 1.3204 and 1.3282

$$x_6 = \frac{1.3204 + 1.3282}{2} = 1.3243$$

$f(x_6) = f(1.3243) = -ve$

Hence the root lies between 1.3243 and 1.3282

$$x_7 = \frac{1.3243 + 1.3282}{2} = 1.3263$$

$f(x_7) = f(1.3263) = +ve$

$\therefore$ The root lies between 1.3243 and 1.3263

$$x_8 = \frac{1.3243 + 1.3263}{2} = 1.3253$$

$f(x_8) = f(1.3253) = +ve$

The root lies between 1.3243 and 1.3253

$$x_9 = \frac{1.3243 + 1.3253}{2} = 1.3248$$

$f(x_9) = f(1.3248) = +ve$

The root lies between 1.3243 and 1.3248

$$x_{10} = \frac{1.3243 + 1.3248}{2} = 1.32455$$

$f(x_{10}) = f(1.32455) = -ve$

The root lies between 1.3248 and 1.32455

$$x_{11} = \frac{1.3248 + 1.32455}{2} = 1.3247$$

$f(x_{11}) = f(1.3247) = -ve$

$\therefore$ The root lies between 1.3247 and 1.3248

Hence, $x_{12} = \dfrac{1.3247 + 1.3248}{2} = 1.32475$

Therefore, the approximate root is 1.32475

3.2 Iteration method (or method of successive approximations)

Suppose we want the approximate roots of the equation

$$f(x) = 0$$

Now, write the equation (1) in the form

$$x = \varphi(x)$$

Assume x_0 to be the starting approximate value to the actual root α of $x = \varphi(x)$. Setting $x = x_0$ in the right hand side of (2), we get first approximation

$$x_1 = \varphi(x_0)$$

Again setting $x = x_1$ on the R.H.S of (2), we get successive approximations

$$x_2 = \varphi(x_1)$$

$$x_3 = \varphi(x_2)$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$x_n = \varphi(x_{n-1})$$

The sequence of approximate roots $x_1, x_2 \ldots x_n$ if it converges to α is taken as the root of the equation $f(x) = 0$.

3.2.1 The condition for the convergence of the method

<u>Theorem</u>: Let $f(x) = 0$ be the given equation whose actual root is α. The equation $f(x) = 0$ be written as $x = \varphi(x)$. Let I be the interval containing the root $x = \alpha$. If $|\varphi'(x)| < 1$ for all x in I, then the sequence of approximation $x_0, x_1, x_2 \ldots, x_n$ will converge to α, if the initial starting value x_0 is chosen in I

Proof: Since α is an actual root of $x=\varphi(x)$, we have

$$\alpha = \varphi(\alpha)$$
$$x_1 = \varphi(x_0)$$
$$x_2 = \varphi(x_1)$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$x_n = \varphi(x_{n-1})$$

from which the sequence $x_0, x_1, x_2 \ldots, x_n$ of approximations is got. Hence, $$x_n - \alpha = \varphi(x_{n-1}) - \varphi(\alpha)$$

By mean value theorem of differential calculus,

$$\varphi(x_{n-1}) - \varphi(\alpha) = (x_{n-1}-\alpha)\,\varphi'(\theta) \quad \text{where } x_{n-1} < \theta < \alpha$$

Using in (4),

$$x_n - \alpha = (x_{n-1} - \alpha)\,\varphi'(\theta)$$

Let $|\varphi'(x)| \leq k$ for all x in the interval I which contains

$$x_0, x_1, x_2 \ldots x_n, \alpha$$

Hence, (5) reduces to,

$$|x_n - \alpha| \leq |x_{n-1} - \alpha|\,k$$

Similarly $|x_{n-1} - \alpha| \leq |x_{n-2} - \alpha|\,k$

$$|x_{n-2} - \alpha| \leq |x_{n-3} - \alpha|\,k$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$|x_1 - \alpha| \leq |x_0 - \alpha|\,k$$

Multiplying vertically and cancelling the factors,

$$|x_n - \alpha| \leq k^n\,|x_0 - \alpha|$$

If $\quad k<1,\ k^n \to 0$ as $n \to \infty$

Hence $\quad |x_n - \alpha| \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} x_n = \alpha$$

Therefore, the sequence of approximations $x_0, x_1, \ldots x_n$ converges to the exact root α if

$|\varphi(x)| < k < 1$ for all values of x in I. The sequence will converge rapidly if $|\varphi'(x)|$ is very small

If $|\varphi'(x)| > 1$, $|x_n - \alpha|$ will become very great and the sequence will not converge.

6.2 Solve $e^x - 3x = 0$ by the method of iteration

Solution. Let $f(x) = e^x - 3x = 0$

$f(0) = 1 = +ve$; $f(1) = e - 3 = -ve$

$\therefore$ a root lies between 0 and 1

Let $x = \frac{1}{3}e^x = \varphi(x)$

$\varphi'(x) = \frac{1}{3}e^x$ and $\varphi'(0) = \frac{1}{3}$, $\varphi'(1) < 1$

In the iterval $(0,1)$, $|\varphi'(x)| < 1$

Select $x_0 = 0.6$, $x_1 = \frac{1}{3}e^{x_0} = \frac{1}{3}e^{0.6} = 0.60737$

$x_2 = \frac{1}{3}e^{0.60737} = 0.61187$, $x_3 = \frac{1}{3}e^{0.61187} = 0.61452$

$x_4 = \frac{1}{3}e^{0.61452} = 0.61626$, $x_5 = \frac{1}{3}e^{0.61626} = 0.61733$

$x_6 = \frac{1}{3}e^{0.61733} = 0.61799$, $x_7 = \frac{1}{3}e^{0.61799} = 0.61840$

$x_8 = \frac{1}{3}e^{0.61840} = 0.61865$, $x_9 = \frac{1}{3}e^{0.61865} = 0.61881$

$x_{10} = \frac{1}{3}e^{0.61881} = 0.61891$, $x_{11} = \frac{1}{3}e^{0.61891} = 0.61897$

$x_{12} = \frac{1}{3}e^{0.61897} = 0.61900$, $x_{13} = \frac{1}{3}e^{0.61900} = 0.61902$

We can take 0.6190 as the correct value of the root of the equation.

6.3 Regula Falsi method (or the method of false position)

Consider the equation $f(x) = 0$ and let $f(a)$ and $f(b)$ be of opposite signs. Also, let $a < b$. The curve $y = f(x)$ will meet the x-axis at some point between $A(a, f(a))$ and $B(b, f(b))$. The equation of the chord joining the two points $A(a, f(a))$ and $B(b, f(b))$ is $\frac{y - f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$.

The x-coordinate of the point of intersection of this chord with the x-axis given an approximate value

(8)

for the root of $f(x)=0$. Setting $y=0$ in the chord equation, we get

$$\frac{-f(a)}{x-a} = \frac{f(a)-f(b)}{a-b}.$$

$$x[f(a)-f(b)] - af(a) + af(b) = -af(a) + bf(a)$$

$$x[f(a)-f(b)] = bf(a) - af(b)$$

$$\therefore x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

The value of x_1 given an approximate value of the root of $f(x)=0$. $(a<x_1<b)$

Now $f(x_1)$ and $f(a)$ are of opposite sign or $f(x_1)$ and $f(b)$ are of opposite signs

If $f(x_1)\, f(a) < 0$, then x_2 lies between x_1 and a

Hence $x_2 = \dfrac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)}$

In the same way, we get $x_3, x_4, \ldots$

This sequence $x_1, x_2, x_3, \ldots$ will converge to the required root. In practice, we get x_i and x_{i+1} such that $|x_i - x_{i+1}| < \epsilon$, the required accuracy.

Find the positive root of $x^3 = 2x+5$ by False position method.

Solution. Let $f(x) = x^3 - 2x - 5 = 0$

There is only one positive root by Descarte's rule of signs.

$$f(2) = 8-9 = -1 = -ve \;;\; f(3) = 16 = +ve$$

$\therefore$ The positive root lies between 2 and 3.

It is closer to 2 also

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{2\times f(3) - 3f(2)}{f(3)-f(2)}$$

$$= \frac{32+3}{17} = 2.058824$$

$$f(x_1) = f(2.058824) = -0.390795$$

$\therefore$ The root lies between 2.058824 and 3

$$x_2 = \frac{2.058824 \times f(3) - 3 \times f(2.058824)}{f(3) - f(2.058824)}$$

$$= \frac{34.113569}{16.390795} = 2.081264$$

$f(x_2) = f(2.081264) = -0.147200$

$\therefore$ The root lies between 2.081264 and 3

$$x_3 = \frac{2.081264 \times 16 - 3 \times (-0.147200)}{16 + 0.147200} = 2.089639$$

$f(x_3) = f(2.089639) = -0.054679$

The root lies between 2.089639 and 3

$$\therefore x_4 = \frac{2.089639 \times f(3) - 3 \times f(2.089639)}{f(3) - f(2.089639)} = 2.092740$$

$f(x_4) = f(2.09274) = -0.020198$

$\therefore$ The root lies between 2.09274 and 3

$$x_5 = \frac{2.09274 \times 16 + 3 \times (0.020198)}{16.020198} = 2.093884$$

$f(x_5) = f(2.093884) = -0.007447$

The root lies between 2.093884 and 3

$$x_6 = \frac{2.093884 \times 16 + 3 \times 0.007447}{16.007447} = 2.0943$$

$f(x_6) = f(2.094306) = -0.002740$

$\therefore$ The root lies between 2.094306 and 3

$$x_7 = \frac{2.094306 \times 16 - 3 \times (-0.002740)}{16.002740} = 2.0$$

Similarly, $x_8 = 2.0945$ correct to 4 decimal places.

3.4 Newton - Raphson method (or Newton's method)

Given an approximate value of a root of equation, a better and closer approximation to the r can be found by using an interative process called Newton's method or Newton - Raphson method.

Let α_0 be an approximate value of a root of the equation $f(x)=0$

Let α be the exact root nearer to α_0

Then $\alpha = \alpha_0 + h$ where h is very small, positive or negative.

$\therefore f(\alpha) = f(\alpha_0 + h) = 0$ since α is the exact root of $f(x)=0$ By Taylor expansion

$$f(\alpha) = f(\alpha_0 + h) = f(\alpha_0) + h f'(\alpha_0) + \frac{h^2}{2!} f''(\alpha_0) + \dots = 0$$

i.e, If h is small, neglecting $h^2, h^3, \dots$ etc, we get

$$f(\alpha_0) + h f'(\alpha_0) = 0$$

$$\therefore h = -\frac{f(\alpha_0)}{f'(\alpha_0)} \text{ if } f'(\alpha_0) \neq 0$$

$$\therefore \alpha = \alpha_0 + h = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)} \text{ approximately}$$

Let this value be α_1

$$\therefore \alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$$

α_1 is a better approximate root than α_0

starting with this α_1, we get

$$\alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)} \text{ which is still better}$$

Continuing like this, we iterate this process until $|\alpha_{r+1} - \alpha_r|$ is less than the quantity desired

$$\therefore \alpha_{r+1} = \alpha_r - \frac{f(\alpha_r)}{f'(\alpha_r)}, \quad r = 0, 1, 2, \dots$$

This is the iterative formula of Newton-Raphson method.

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4 Using newton's method find the positive root lies between 0 and 1 of $x^3 = 6x - 4$ Correct to 5 decimal places.

Soln: $f(x) = x^3 - 6x + 4 = 0$; $f'(x) = 3x^2 - 6$

$f(0) = 4 = +ve$

$f(1) = 1 - 6 + 4 = -1 = -ve$

The root lies between $(0,1)$

$$x_0 = \frac{0+1}{2} = 0.5$$

<u>Formula</u>

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

put $r=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{1.125000}{-5.25000} = 0.5 + \frac{1.12500}{5.25000}$$

$$= 0.5 + 0.21429 = 0.71429$$

$$\boxed{x_1 = 0.71429}$$

put $r=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.71429 + \frac{0.07870}{4.46937} = 0.71429 + 0.017$$

$$\boxed{x_2 = 0.73190}$$

put $r=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.73190 + \frac{0.00066}{4.39297}$$

$$\boxed{x_3 = 0.73205}$$

put $r=3$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.73205 - 0.0000$$

$$\boxed{x_4 = 0.73205}$$

Hence the approximate root is 0.73205

UNIT - II

Solution of Simultaneous linear Algebraic Equations

4.2 Gauss - Elimination method (Direct method).

This is a direct method based on the eliminati[on] of the unknowns by combining equations such that th[e] n equations in n unknowns are reduced to an equival[ent] upper triangular system which could be solved by back substitution.

Consider the n linear equations in n unknow[ns] viz

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \quad \text{---}\to (1)$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where a_{ij} and b_i are known constants and x_i's are unknow[n]

The system (1) is equivalent to

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ ....

Now our aim is to reduce the augmented matri[x] (A,B) to upper triangular matrix

$$(A,B) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \quad \cdots\to (3)$$

Now, multiply the first row of (3) (if $a_{11} \neq 0$) $-\frac{a_{i1}}{a_{11}}$ and add to the i^{th} row of (A,B), where $i = 2,3\ldots n$

By this, all elements in the first column of (A, B) except a_{11} are made to zero. Now (3) is of the form

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & \cdots & b_{2n} & c_2 \\ \cdot & \cdots & \cdots & \cdots & \cdot \\ 0 & b_{n2} & \cdots & b_{nn} & c_n \end{array}\right] \quad \cdots\to(4)$$

Now take the pivot b_{22}. Now, considering b_{22} as the pivot, we will make all elements below b_{22} in the second column of (4) as zeros. That is, multiply second row of (4) by $-\frac{b_{i2}}{b_{22}}$ and add to the corresponding elements of the ith row $(i = 3, 4, \ldots n)$. Now all elements below b_{22} are reduced to zero.

Now (4) reduces to

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} & c_2 \\ 0 & 0 & c_{33} & \cdots & c_{3n} & d_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \\ 0 & 0 & c_{n3} & \cdots & c_{nn} & d_n \end{array}\right] \quad \cdots\to(5)$$

Now taking c_{33} as the pivot, using elementary operations, we make all elements below c_{33} as zeros. Continuing the process, all elements below the leading diagonal elements of A are made to zero.

Hence, we get (A, B) after all these operations

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & & \cdots & b_{2n} & c_2 \\ 0 & 0 & c_{33} & c_{34} & \cdots & c_{3n} & d_3 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & d_{nn} & k_n \end{array}\right] \quad \cdots\to(6)$$

From (6), the given system of linear equations is equivalent to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$b_{22}x_2 + b_{23}x_3 + \cdots + b_{2n}x_n = c_2$$

$$c_{33}x_3 + \cdots c_{3n}x_n = d_3$$
$$\cdots\cdots\cdots$$
$$d_{nn}x_n = k_n$$

Going from the bottom of these equation, we solve for $x_n = \frac{k_n}{d_{nn}}$ Using this in the penultimate equation, we get x_{n-1} and so. By this back substitution method we solve for

$$x_n, x_{n-1}, x_{n-2}, \cdots x_2, x_1.$$

4.2.1 Gauss - Jordan elimination method (Direct method)

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system $AX=B$ is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making all elements above the leading diagonal of A also as zeros. By this way, the system $AX=B$ will reduce to the form.

$$\left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & 0 & b_1 \\ 0 & b_{22} & 0 & 0 & 0 & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & d_3 \\ 0 & 0 & 0 & 0 & d_{nn} & k_n \end{array}\right]$$

From (7)

$$x_n = \frac{k_n}{d_{nn}}, \cdots x_2 = \frac{c_2}{b_{22}}, x_1 = \frac{b_1}{a_{11}}$$

1 Solve the system of equations by (i) Gauss eliminatio method (ii) Gauss - Jordan method

$$x+2y+z = 3, \quad 2x+3y+3z = 10, \quad 3x-y+2z = 13$$

Solution. (By Gauss elimination method)

The given system is equivalent to

$$\underset{A}{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{B}{\begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}}$$

$$(A, B) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$$

Now, we will make the matrix A upper triangular

$$(A, B) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right] \quad \begin{array}{l} R_2 + (-2)R_1 \text{ ie, } R_{21}(-2) \\ R_3 + (-3)R_1 \text{ ie, } R_{31}(-3) \end{array}$$

Now take $b_{22} = -1$ as the pivot and make b_{32} as zero.

$$(A, B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad R_{32}(-7)$$

From this, we get

$$x + 2y + z = 3$$
$$-y + z = 4$$
$$-8z = -24$$

$\therefore$ $z = 3$, $y = -1$, $x = 2$ by back substitution

ie., $x = 2$, $y = -1$, $z = 3$.

Solution (Gauss - Jordan method)

In stage 2, make the element, in the position $(1,2)$, also zero.

$$(A, B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad R_{12}(2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right] R_3\left(\tfrac{1}{8}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right] R_{13}(3), R_{23}(1)$$

ie, $x = 2, \; -y = 1, \; -z = -Z.$

ie, $x = 2, \; y = -1, \; Z = 3$

4.7 Iterative methods.

All the previous methods seen in solving the system of simultaneous algebraic linear equations are direct methods. Now we will see some indirect methods or iterative methods.

This iterative methods is not always successful to all systems of equations. If this method is to succeed, each equation of the system must possess one large coefficient and the large coefficient must be attached to a different unknown in that equation. This condition will be satisfied if the large coefficients are along the leading diagonal of the coefficient matrix. when the condition is satisfied, the system will be solvable by the iterative method. The system,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Will be solvable by this method if

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

In other words, the solution will exist if the absolute values of leading diagonal elements of the coefficient matrix A of the system $AX = B$ are greater than the sum of absolute values of the other coefficients of that row. The condition is sufficient but not necessary.

4.8 Jacobi method of iteration or Gauss-Jacobi method.

Let us explain this method in the case of three equations in three unknowns.

Consider the system of equations.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \quad \ldots\ldots \to (1)$$

Let us assume $|a_1| > |b_1| + |c_1|$

$|b_2| > |a_2| + |c_2|$

$|c_3| > |a_3| + |b_3|$

Then, iterative method can be used for the system (1) That is solve for x, y, z in terms of the other variables,

$$\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned} \quad \ldots\ldots \to (2)$$

If $x^{(0)}, y^{(0)}, z^{(0)}$ are the initial values of x, y, z respectively, then

$$\begin{aligned} x^{(1)} &= \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}) \\ y^{(1)} &= \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)}) \\ z^{(1)} &= \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)}) \end{aligned} \quad \ldots\ldots \to (3)$$

Again using these values $x^{(1)}, y^{(1)}, z^{(1)}$ in (2), we get

$$\begin{aligned} x^{(2)} &= \frac{1}{a_1}(d_1 - b_1y^{(1)} - c_1z^{(1)}) \\ y^{(2)} &= \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(1)}) \\ z^{(2)} &= \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)}) \end{aligned} \quad \ldots\ldots \to (4)$$

Proceeding in the same way, if the r^{th} iterates are $x^{(r)}, y^{(r)}, z^{(r)}$, the interation scheme reduces to

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r)} - c_2 z^{(r)}) \quad \ldots\ldots \to (5)$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r)} - b_3 y^{(r)})$$

The procedure is continued till the convergence is assured

4.9 Gauss - seidal method of iteration

This is only a refinement of Gauss-Jacobi method. As before,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \quad \ldots\ldots \to (6)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We start with the initial values $y^{(0)}, z^{(0)}$ for y and z and get $x^{(1)}$ from the first equation. That is

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

While using the second equation, we use $z^{(0)}$ for z an $x^{(1)}$ for x instead of $x^{(0)}$ as in the jacobi's method, we ge

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)}$ and $y^{(1)}$, use $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side.

If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

This process of iteration is continued until the convergence assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss-Seidal method is very fast when compared to Gauss-Jacobi method. The rate of convergence in Gauss-Seidal method is roughly two times than that of Gauss-Jacobi method. As we saw the sufficient conditions already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is the method of iteration will converge if in each equation of the given system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients.

Solve the following system by Gauss-Jacobi and Gauss-Seidal methods.

$10x - 5y - 2z = 3$; $4x - 10y + 3z = -3$; $x + 6y + 10z = -3$

Solution. Here, we see that the diagonal elements are dominant. Hence, the iteration process can be applied

That is, the coefficient matrix $\begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$ is diagonally dominant, since

$|10| > |-5| + |-2|$, $|-10| > |4| + |3|$ and $|10| > |1| + |6|$

Gauss-Jacobi method. Solving for x, y, z we have

$$x = \frac{1}{10}(3 + 5y + 2z) \quad \ldots \to (1)$$

$$y = \frac{1}{10}(3 + 4x + 3z) \quad \ldots \to (2)$$

$$z = \frac{1}{10}(-3 - x - 6y) \quad \ldots \to (3)$$

First iteration: Let the initial values be $(0,0,0)$ using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}[3 + 5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3+4(0)+3(0)] = 0.3$$

$$z^{(1)} = \frac{1}{10}[-3-(0)-6(0)] = -0.3$$

Second iteration : Using these values in (1), (2), (3) we get

$$x^{(2)} = \frac{1}{10}[3+5(0.3)+2(-0.3)] = 0.39$$

$$y^{(2)} = \frac{1}{10}[3+4(0.3)+3(-0.3)] = 0.33$$

$$z^{(2)} = \frac{1}{10}[-3-(0.3)-6(0.3)] = -0.51$$

Third iteration : Using the values of $x^{(2)}, y^{(2)}, z^{(2)}$ in (1),(2),(3) we ge

$$x^{(3)} = \frac{1}{10}[3+5(0.33)+2(-0.51)] = 0.363$$

$$y^{(3)} = \frac{1}{10}[3+4(0.39)+3(-0.51)] = 0.303$$

$$z^{(3)} = \frac{1}{10}[-3-(0.39)-6(0.33)] = -0.537$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}[3+5(0.303)+2(-0.537)] = 0.3441$$

$$y^{(4)} = \frac{1}{10}[3+4(0.363)+3(-0.537)] = 0.2841$$

$$z^{(4)} = \frac{1}{10}[-3-0.363-6(0.303)] = -0.5181$$

Fifth iteration :

$$x^{(5)} = \frac{1}{10}(3+5(0.2841)+2(-0.5181)] = 0.33843$$

$$y^{(5)} = \frac{1}{10}[3+4(0.3441)+3(-0.5181)] = 0.2822$$

$$z^{(5)} = \frac{1}{10}[-3-(0.3441)-6(0.2841)] = -0.50487$$

Sixth iteration :

$$x^{(6)} = \frac{1}{10}[3+5(0.2822)+2(-0.50487)] = 0.340126$$

$$y^{(6)} = \frac{1}{10}[3+4(0.33843)+3(-0.50487)] = 0.283911$$

$$z^{(6)} = \frac{1}{10}[-3-(0.33843)-6(0.2822)] = -0.503163$$

Seventh iteration :

$$x^{(7)} = \frac{1}{10}[3+5(0.283911)+2(-0.503163)] = 0.3413229$$

$$y^{(7)} = \frac{1}{10}[3+4(0.340126)+3(-0.503163)] = 0.2851015$$

$$z^{(7)} = \frac{1}{10}[-3-(0.340126)-6(0.283911)] = -0.5043592$$

Eighth iteration :

$$x^{(8)} = \frac{1}{10}[3+5(0.2851015)+2(-0.5043592)] = 0.3416789$$

$$y^{(8)} = \frac{1}{10}[3+4(0.3413229)+3(-0.5043592)] = 0.2852214$$

$$z^{(8)} = \frac{1}{10}[-3-(0.3413229)-6(0.2851015)] = -0.50519319$$ (10)

Nineth iteration :

$$x^{(9)} = \frac{1}{10}[3+5(0.2852214)+2(-0.50519319)] = 0.341572062$$

$$y^{(9)} = \frac{1}{10}[3+4(0.34167891)+3(-0.50519319)] = 0.285113607$$

$$z^{(9)} = \frac{1}{10}[-3-(0.34167891)-6(0.2852214)] = -0.505300731$$

Hence correct to 3 decimal places, the value are

$x = 0.342$, $y = 0.285$, $z = -0.505$

Gauss - seidal method : Initial values : $y=0, z=0$

First iteration :

$$x^{(1)} = \frac{1}{10}[3+5(0)+2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3+4(0.3)+3(0)] = 0.42$$

$$z^{(1)} = \frac{1}{10}[-3-(0.3)-6(0.42)] = -0.582$$

Second iteration :

$$x^{(2)} = \frac{1}{10}[3+5(0.42)+2(-0.582)] = 0.3936$$

$$y^{(2)} = \frac{1}{10}[3+4(0.3936)+3(-0.582)] = 0.28284$$

$$z^{(2)} = \frac{1}{10}[-3-(0.3936)-6(0.28284)] = -0.509064$$

Third iteration :

$$x^{(3)} = \frac{1}{10}[3+5(0.28284)+2(-0.509064)] = 0.3396072$$

$$y^{(3)} = \frac{1}{10}[3+4(0.3396072)+3(-0.509064)] = 0.2831236$$

$$z^{(3)} = \frac{1}{10}[-3-(0.3396072)-6(0.28312368)] = -0.50383\,928$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}[3+5(0.28312368)+2(-0.50383492)] = 0.34079$$

$$y^{(4)} = \frac{1}{10}[3+4(0.34079485)+3(-0.50383492)) = 0.2851674$$

$$z^{(4)} = \frac{1}{10}[-3-(0.34079485)-6(0.28516746)) = -0.50517$$

Fifth iteration :

$$x^{(5)} = \frac{1}{10}[3+5(0.28516746)+2(-0.50517996)] = 0.34155477$$

$$y^{(5)} = \frac{1}{10}[3+4(0.34155477)+3(-0.50517996)] = 0.28506792$$

$$z^{(5)} = \frac{1}{10}[-3-(0.34155477)-6(0.28506792)] = -0.505196229$$

Sixth iteration :

$$x^{(6)} = \frac{1}{10}[3+5(0.28506792)+2(-0.505196229)] = 0.341494714$$

$$y^{(6)} = \frac{1}{10}[3+4(0.341494714)+3(-0.505196229)] = 0.285039017$$

$$z^{(6)} = \frac{1}{10}[-3-(0.341494714)-6(0.285039017)] = -0.5051728$$

Seventh iteration :

$$x^{(7)} = \frac{1}{10}[3+5(0.285039017)+2(-0.5051728)] = 0.3414849$$

$$y^{(7)} = \frac{1}{10}[3+4(0.3414849)+3(-0.5051728)] = 0.28504212$$

$$z^{(7)} = \frac{1}{10}[-3-(0.3414849)-6(0.28504212)] = -0.5051737$$

The values correct to 3 decimal places are

$x = 0.342$, $y = 0.285$, $z = -0.505$

UNIT - III

Difference Equations

Definition: 10.1

An equation which expresses a relation between the independent variable, the dependent variable and the successive difference of the dependent variable is called a difference equation

Eg:-

$$\Delta^3 y_x - 4\Delta y_x + 7 y_x = x^2 + \cos x + 7 \longrightarrow (1)$$

$\Delta^2 y_x - 2\Delta y_x + y_x = 0$ are difference equations.

using $\Delta = E-1$, $\Delta^n = (E-1)^n$ we can write

$$\Delta y_x = (E-1) y_x = E y_x - y_x = y_{x+1} - y_x$$

$$\Delta^2 y_x = (E-1)^2 y_x = y_{x+2} - 2y_{x+1} + y_x \text{ etc}$$

Hence $\Delta^2 y_x - 2\Delta y_x + 3y_x = x^2$ can be written as

$$y_{x+2} - 4y_{x+1} + 6y_x = x^2 \longrightarrow (2)$$

$$\text{(or) } y(x+2) - 4y(x+1) + 6y(x) = x^2 \longrightarrow (3)$$

$$\text{(or) } E^2 y_x - 4E y_x + 6y_x = x^2$$

$$(E^2 - 4E + 6) y_x = x^2 \longrightarrow (4)$$

This indicates that a difference equation can be written in various forms such as (1), (2), (3) and (4).

10.2. Order and degree of a difference equation

Order :-

The order of a difference equation written in the form free form Δ's is the difference between the highest and lowes subscripts of y on arguments of y.

eg: $y_{x+3} - 5y_{x+2} + 7y_{x+1} + y_x = 10x$

highest subscribt $= x+3$

Lowest subscribt $= x$

order = highest subscribt − Lowest subscribt

$= \cancel{x}+3-\cancel{x}$

order $= 3$.

DEGREE : Definition

The degree of a difference equation written in a form free form Δ's in the highest power of y.

eg:- $y_{x+1} \, y^5_{x+2} - y_{x+1} y_x + y^2_{x+3} = \cos x$

Degree $= 5$

10.3 Linear difference equations

An equation of the form

$a_0 y_{x+n} + a_1 y_{x+n-1} + a_2 y_{x+n-2} + \dots + a_{n-1} y_{x+1} + a_0 y_x = \varphi(x)$

$\left[a_0 E^n + a_1 E^{n-1} + a_2 E^{n-2} + \dots + a_{n-1} E + a_n\right] y_x = \varphi(x) \longrightarrow$

Where $a_0, a_1, a_2, \dots a_n$ and $\varphi(x)$ are known functions of x. there is called a linear Difference equation in y_x

Homogeneous

The eqn ① or ② can be written as $f(E)y_x = \varphi(x) \rightarrow ③$

where $f(E)$ is a polynomial express in E.

The Right side of an equation ③ $\varphi(x)$ is zero then

$f(E)y_x = 0 \longrightarrow ④$ is called the homogeneous equation corresponding equation ③

Non - Homogeneous.

The solution of the non-homogeneous linear equation ③ depends upon the corresponding homogeneous Linear equation.

10.4 Find the complementary function of $f(E)y_x = \varphi(x)$ ...①

Soln :- Let $f(E)y_x = \varphi(x)$ be the linear eqn write down the Auxiliary equation Replace E by ai $f(E)y_x = \varphi(x)$

$$f(a) = 0$$

we get the roots $a_0, a_1, \ldots a_n$

Case (i):- If the roots $a_0, a_1, \ldots a_n$ are all real and distinct the corresponding complementary function of eqn ① are complete solution of $f(E)y_x = 0$ is

$$y_x = c_1 a_1^x + c_2 a_2^x + \cdots + c_n a_n^x.$$

Case (ii) :- If the roots $a_1 = a_2$ the corresponding complementary function.

$$y_x = (c_1 + c_2 x) a_1^x + c_3 a_3^x + \cdots + c_n a_n^x$$

Case (iii) :

imaginary, Distinct

If $a_1 = \alpha + i\beta$. $a_2 = \alpha - i\beta$

Then $y_x = A(\alpha + i\beta)^x + B(\alpha - i\beta)^x + c_3 a_3^x + \cdots + c_n a_n^x$

$$y_x = A[r(\cos\theta + i\sin\theta)]^x + B[r(\cos\theta - i\sin\theta)]^x + c_3 a_3^x + \dots + c_n a_n^x$$

$$y_x = r^x[c_1\cos\theta x + c_2\sin\theta x] + c_3 a_3^x + \dots + c_n a_n^x$$

where $r = |\alpha + i\beta|$; $\theta = \text{amp}(\alpha + i\beta) = \tan^{-1}\frac{\beta}{\alpha}$

Case :-

If $a_1 = a_2 = \alpha + i\beta$

$a_3 = a_4 = \alpha - i\beta$

then the complete solution of $f(E)y_x = 0$ is

$$y_x = r^x[(c_1 + c_2 x)\cos\theta x + (c_3 + c_4 x)\sin\theta x] + c_5 a_5^x + \dots + c_n a_n^x$$

Ex: Form the Difference equation of lowest order by eliminating the arbitrary constants a & b

$y = a\cdot 2^x + b\cdot 3^x$

solution:

$$y_x = a\cdot 2^x + b\cdot 3^x \longrightarrow (1)$$

$$y_{x+1} = a\cdot 2^{x+1} + b\cdot 3^{x+1} \longrightarrow (2)$$

$$y_{x+2} = a\cdot 2^{x+2} + b\cdot 3^{x+2} \longrightarrow (3)$$

(1) × 2 − (2)

$$2y_x = a\cdot 2^{x+1} + 2b\cdot 3^x$$

$$\overset{(-)}{}y_{x+1} = \overset{(-)}{}a\cdot 2^{x+1} \overset{(-)}{+} b\cdot 3^{x+1}$$

$$2y_x - y_{x+1} = 2b3^x + b3^x\cdot 3$$

$$2y_x - y_{x+1} = b3^x(2-3)$$

$$2y_x - y_{x+1} = -b3^x \longrightarrow (4)$$

(2) × 2 − (3)

$$2y_{x+1} = a\cdot 2^{x+2} + 2b3^{x+1}$$

$$\overset{(-)}{}y_{x+2} = \overset{(-)}{}a\,2^{x+2} \overset{(-)}{+} b\cdot 3^{x+2}$$

$$2y_{x+1} - y_{x+2} = 2b3^{x+1} - b3^{x+1}\cdot 3$$

$$2y_{x+1} - y_{x+2} = b3^{x+1}(2-3)$$

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$$2y_{x+1} - y_{x+2} = -b\,3^{x+1} \longrightarrow (5)$$

$$\frac{(4)}{(5)} \Rightarrow \frac{2y_x - y_{x+1}}{2y_{x+1} - y_{x+2}} = \frac{-b\,3^x}{-b\,3^{x+1}}$$

$$3(2y_x - y_{x+1}) = 2y_{x+1}$$

$$6y_x - 3y_{x+1} - 2y_{x+1} + y_{x+2} = 0$$

$$y_{x+2} - 5y_{x+1} + 6y_x = 0$$

Ex: 2 Form the difference equation given

$$y_n = (An+B)\,3^n$$

Solution:

$$y_n = (An+B)\,3^n \longrightarrow (1)$$

$$y_{n+1} = [A(n+1)+B]\,3^{n+1}$$

$$\tfrac{1}{3}\,y_{n+1} = (An+A+B)\,3^n \longrightarrow (2)$$

$$y_{n+2} = [A(n+2)+B]\,3^{n+2}$$

$$= [An+2A+B]\,3^n \cdot 3^2$$

$$\tfrac{1}{9}\,y_{n+2} = [An+2A+B]\,3^n \longrightarrow (3)$$

$(1) - 2(2) + 3$:-

$$y_n + \tfrac{1}{9}y_{n+2} - \tfrac{2}{3}y_{n+1}$$

$$= An3^n + B3^n + An3^n + 2A3^n + B3^n - 2(An3^n + A3^n + B$$

$$9y_n + y_{n+2} - 6y_{n+1} = 0$$

$$\therefore\ y_{n+2} - 6y_{n+1} + 9y_n = 0$$

10.6 To find particular integral of $f(E)\, y_x = \varphi(x)$ $\longrightarrow (1)$

Type 1. $f(E)\, y_x = a^x$, where a is constant

$$\frac{a^x}{f(E)} = \frac{a^x}{f(a)} \text{ if } f(a) \neq 0$$

If $f(a) = 0$

$$\frac{a^x}{f(E)} = \frac{a^x}{(E-a)\,\varphi(x)}$$

$$\frac{a^x}{(E-a)^2} = \frac{x^{(2)}}{2!}\, a^{x-2}$$

$$\frac{a^x}{(E-a)^r} = \frac{x^{(r)}}{r!}\, a x^{x-r}$$

Ex: solve $y_{n+2} - 4y_{n+1} + 3y_n = 2^n + 3^n + 7$.

Solution: Given $(E^2 - 4E + 3)\, y_n = 2^n + 3^n + 7$

Auxiliary equation

$$a^2 - 4a + 3 = 0$$

$$a = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2}$$

$$a = 2 \pm 1 \quad \boxed{a_1 = 1}, \quad \boxed{a_2 = 3}$$

The roots are real and Distinct

$\therefore$ Complementary function

$$\boxed{y_x = c_1 1^x + c_2 3^x}$$

$$P.I_1 = \frac{2^n}{E^2 - 4E + 3} = \frac{2^n}{(E-1)(E-3)} = \frac{2^n}{(2-1)(2-3)} = -2^n$$

$$P.I_2 = \frac{3^n}{(E-1)(E-3)} = \frac{3^n}{(E-3)(3-1)} = \frac{3^n}{(E-3)(2)} = \frac{1}{2} n\, 3^{n-1}$$

$$P.I_3 = \frac{7 \cdot 1^n}{(E-1)(E-3)} = \frac{7 \cdot 1^n}{-2(E-1)} = -\frac{7}{2} n 1^{n-1}$$

$$y = C.F + P.I_1 + P.I_2 + P.I_3$$

$$y = c_1 + c_2 3^x - 2^n + \frac{n\,3^{n-1}}{2} - \frac{7}{2}n$$

Type - II

let $\phi(x)$ = a polynomial in x of degree m, then

P.I $= \frac{\phi(x)}{f(E)} = [f(1+\Delta)]^{-1}\phi(x)$ we expand $f(x)[f(1+\Delta)]^{-1}$ is ascending follows of Δ and then operate on $\phi(x)$

Solve $y_{x+2} - 4y_x = 9x^2$

Solution: $(E^2 - 4)\,y_x = 9x^2$.

Auxiliary equation is

$a^2 - 4 = 0 \Rightarrow a^2 = 4 \Rightarrow a = \sqrt{4} \Rightarrow a = \pm 2$

$\therefore a_1 = +2;\ a_2 = -2$

The roots are real and distinct

$$\boxed{C.F = c_1 2^x + c_2(-2)^x}$$

$$P.I = \frac{9x^2}{E^2 - 4}$$

$$= \frac{9x^2}{(1+\Delta)^2 - 4} = \frac{9x^2}{1+\Delta^2+2\Delta-4} = \frac{9x^2}{\Delta^2+2\Delta-3}$$

$$= \frac{{}^{3}\cancel{9}x^2}{-\cancel{3}\left[1 - \frac{\Delta^2+2\Delta}{3}\right]} = 3\left[1 - \left(\frac{\Delta^2+2\Delta}{3}\right)\right]^{-1} x^2$$

$$= -3\left[1 + \left(\frac{\Delta^2+2\Delta}{3}\right) + \left(\frac{\Delta^2+2\Delta}{3}\right)^2 + \cdots\right]\left(x^{(2)} + x^{(1)}\right)$$

$$= -3\left[1 + \frac{7\Delta^2}{9} + \frac{2\Delta}{3}\right]\left[x^{(2)} + x^{(1)}\right]$$

$$= -3\left[x(x-1) + x + \frac{14}{9} + \frac{4x}{3} + 2/3\right]$$

$$= \frac{-3\left[9x^2 + 12x + 20\right]}{9}$$

$$P.I = -\left[3x^2 + 4x + \frac{20}{3}\right]$$

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$\therefore \quad y_x = C.F + P.I$

$$\boxed{y_x = c_1 2^x + c_2 (-2)^x - 3x^2 + 4x + \frac{20}{3}}$$

Type - III

If R.H.S $\varphi(x) = \cos kx$ (or) $\sin kx$

$\cos kx$ = Real part of e^{ikx}

$\sin kx$ = Imaginary part of e^{iks}

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}$$

$$\sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$$

$$e^{ikx} = \cos kx + i\sin kx$$

Ex: Solve $U(x+1) - a\,u(x) = \cos nx$

Solution: $u(x+1) - a\,u(x) = \cos nx$

shifting operator

$(E-a)\,u(x) = \cos nx$

A.E $\quad m - a = 0 \Rightarrow m = a$

The root is real

$$\boxed{C.F = c_1 a^x}$$

$$P.I = \frac{1}{E-a} \cos nx$$

$$= \text{Real part of } \frac{1}{(E-a)} e^{(in)x} \quad \text{[Replace E}$$

$$= R.p \frac{1}{(e^{in}-a)} e^{(in)x}$$

$$= R.p \frac{e^{inx}}{e^{in}-a} \times \frac{(e^{-in}-a)}{(e^{-in}-a)}$$

$$= R.p \frac{e^{in(x-1)} - ae^{inx}}{1 - ae^{in} - ae^{-in} + a^2}$$

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$$= \frac{\cos n(x-1) - a\cos nx}{1 - a(e^{in}+e^{-in}) + a^2}$$

$$P.I = \frac{\cos n(x-1) - a\cos nx}{1 - 2a\cos n + a^2}$$

$\therefore$ $y = C.F + P.I$

$$y = c_1 a^x + \frac{\cos n(x-1) - a\cos nx}{1 - 2a\cos n + a^2}.$$

Type - IV

Let $\varphi(x) = a^x F(x)$ where $f(x)$ is sum function of

$$\frac{a^x F(x)}{f(E)} = a^x \cdot \frac{F(x)}{f(aE)}$$

Ex:

Solve $u_{n+2} - 7u_{n+1} - 8u_n = 2^n n^2$

Solution Given $[E^2 - 7E - 8]\, u_n = 2^n n^2$

Auxiliary equation is

$$[a^2 - 7a - 8] = 0$$

$$(a+1)(a-8) = 0 \Rightarrow a_1 = -1,\ a_2 = 8$$

$$\boxed{C.F = c_1(-1)^x + c_2\, 8^x}$$

$$P.I = \frac{1}{E^2 - 7E - 8}(2^n \cdot n^2)$$

Replace E by $2E$

$$= 2^n \left[\frac{1}{(2E)^2 - 7(2E) - 8}\right] n^2$$

$$= 2^n \left[\frac{1}{4E^2 - 14E - 8}\right] n^2$$

$$= 2^n \left[\frac{1}{4(1+\Delta)^2 - 14(1+\Delta) - 8}\right] n^2$$

$$= 2^n \left[\frac{1}{4(1+\Delta^2+2\Delta) - 14 - 14\Delta - 8}\right] n^2$$

$$= 2^n\left[\frac{1}{4\Delta^2 - 6\Delta - 18}\right] n^2$$

$$= 2^{n-1} \frac{1}{-9\left[1-\left(\frac{2\Delta^2-3\Delta}{9}\right)\right]} n^2$$

$$= 2^{n-1} \frac{1}{-9}\left[1-\left(\frac{2\Delta^2-3\Delta}{9}\right)\right]^{-1} n^2$$

$$= 2^{n-1} \frac{1}{-9}\left[1+ \frac{2\Delta^2-3\Delta}{9} + \left(\frac{2\Delta^2-\Delta}{9}\right)^2 + \cdots\right] n^{(2)}+n$$

$$= -\frac{2^{n-1}}{9}\left[1+\frac{2\Delta^2}{9} - \frac{\Delta}{3} + \frac{9\Delta^2}{81}\right] n^{(2)}+n^{(1)}$$

$$= -\frac{2^{n-1}}{9}\left[1-\frac{\Delta}{3}+\frac{3\Delta^2}{9}\right](n^{(2)}+n^{(1)})$$

$$= -\frac{2^{n-1}}{9}\left[n^{(2)}+n^{(1)} - \frac{\Delta(n^{(2)}+n^{(1)})}{3} + \frac{\Delta^2(n^{(2)}+n^{(1)}}{3}\right.$$

$$= -\frac{2^{n-1}}{9}\left[n^{(2)}+n^{(1)} - \frac{2n^{(1)}}{3} - \frac{1}{3} + \frac{2}{3}\right]$$

$$= -\frac{2^{n-1}}{9}\left[n(n-1)+n - \frac{2n}{3} + \frac{1}{3}\right]$$

$$= -\frac{2^{n-1}}{9}\left[n^2 - \cancel{n} + \cancel{n} - \frac{2n}{3} + \frac{1}{3}\right]$$

$$P.I = -\frac{2^{n-1}}{9}\left[n^2 - \frac{2n}{3} + \frac{1}{3}\right]$$

$$\therefore y_x = C.F + P.I$$

$$\boxed{y = c(-1)^x + c_2 8^x - \frac{2^{n-1}}{9}\left[n^2 - \frac{2n}{3} + \frac{1}{3}\right]}$$

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UNIT - IV

Numerical solution of ordinary Differential Equations

11.5 Solution by Taylor series (Type 1)

AIM : To find the numerical solution of the equation

$$\frac{dy}{dx} = f(x,y) \quad \cdots (1)$$

given the initial condition $y(x_0) = y_0 \cdots (2)$

Now, we expand $y(x)$ about the point $x = x_0$ in a Taylor's series in powers of $(x-x_0)$. That is

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots \quad \cdots (3)$$

where $y^{(n)}(x_0) = \left(\frac{d^n y}{dx^n}\right)_{x=x_0}$

ie,
$$y(x) = y_0 + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y_0'' + \cdots$$

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad \cdots (4)$$

where $h = x_1 - x_0$ or $x_1 = x_0 + h$

To find $y_0', y_0'' \ldots$ we use (1) and its derivatives at $x = x_0$. Though the series (4) is an infinite series, we can truncate it at any convenient term, if h is small and the accuracy is obtained. Now, having got y_1, We can calculate

$y_1', y_1'', y_1''', \ldots$ etc., by using $y' = f(x, y)$

Now expanding $y(x)$, in a Taylor's series about the point $x = x_1$, we get

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \quad \ldots (5)$$

proceeding in the same way, we get

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y''' + \cdots \quad \ldots (6).$$

Ex: 1 solve $\frac{dy}{dx} = x+y$, given $y(1) = 0$, and get $y(1.1)$, $y(1.2)$ by Taylor series method. Compare your result with the explicit solution.

Solution:

Here $x_0 = 1$, $y_0 = 0$, $h = 0.1$

$y' = x+y$	$y_0 = y(x=1) = 0$
$y'' = 1+y'$	$y_0' = x_0 + y_0 = 1+0 = 1$
$y''' = y''$	$y_0'' = 1 + y_0' = 2$
$y^{iv} = y'''$	$y_0''' = y_0'' = 2$
	$y_0^{iv} = 2$ etc

By Taylor series, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$$

$$\therefore \; y_1 = y(1.1) = 0 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2) + \frac{(0.1)^5}{120}(2) + \cdots$$

$= 0.1 + 0.01 + 0.00033 + 0.00000833 + 0.000000166 + \cdots$

$\boxed{y(1.1) = 0.11033847}$

Now, take $x_0 = 1.1$, $h = 0.1$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \cdots \rightarrow ③$$

We calculate $y_1', y_1'', y_1''', \ldots$; $x_1 = 1.1$, $y_1 = 0.110341833$

$y_1' = x_1 + y_1 = 1.1 + 0.11033847 = 1.21033847$

$y_1'' = 1 + y_1' = 2.21033847$

$y_1''' = y_1'' = y_1^{iv} = y_1^{v} = 2.21033847$

using in ③.

$$y_2 = y(1.2) = 0.11033847 + \frac{0.1}{1!}(1.21033847) + \frac{(0.1)^2}{2}(2.2103384\ldots$$
$$+ \frac{(0.1)^3}{6}(2.21033847) + \frac{(0.1)^4}{24}(2.21033847) + \cdots$$

$\boxed{y_2 = 0.24280160}$

The exact solution of $\frac{dy}{dx} = x + y$ is

$y = -x - 1 + 2e^{x-1}$

$y(1.1) = -1.1 - 1 + 2e^{0.1}$

$= 0.11034$

$y(1.2) = -1.2 - 1 + 2e^{0.2} = 0.2428$

$y(1.1) = 0.11033847$

$y(1.2) = 0.2461077.$

Exact Values : $\boxed{\begin{aligned} y(1.1) &= 0.110341836 \\ y(1.2) &= 0.24280552 \end{aligned}}$

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11.8 Picard's method of successive approximations

AIM. To solve $\frac{dy}{dx} = f(x,y)$ subject to $y(x_0) = y_0$

Now $\frac{dy}{dx} = f(x,y)$ $\longrightarrow$ ①

$\therefore$ $dy = f(x,y)\,dx$

Integrating $y = \int^{x} f(x,y)\,dx + c$ $\longrightarrow$ ②

Setting $x = x_0$ on the R.H.S. after integration and $y = y_0$ on the L.H.S we have

$$y_0 = \int^{x_0} f(x,y)\,dx + c \longrightarrow ③$$

② − ③ $$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\therefore \quad y = y_0 + \int_{x_0}^{x} f(x,y)\,dx \longrightarrow ④$$

In equation ④ the R.H.S integrand $f(x,y)$ involves y also. This type of equation is called integral equation. As the integration is not possible as it is, we will solve it by successive approximation.

Substitute the initial values of y namely y_0 in the integrand $f(x,y)$ in place of y and then integrate the R.H.S to get an approximate value of y on the L.H.S.

ie., $$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx \longrightarrow ⑤$$

Since $f(x,y_0)$ is a function of x also, it is possible to integrate it. w.r.to x

After getting the first approximation $y^{(1)}$ for y use this value $y^{(1)}$ in the place y in $f(x,y)$ of ④ and then integrate to get the second approximation of y namely $y^{(2)}$

ie.,
$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx \longrightarrow ⑥$$

Proceeding in this way, we get the n^{th} approximat value of y as

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx \longrightarrow ⑦$$

Equation ⑦ gives the general iterative formula for y. It is called picard's iteration formula.

Ex: Solve $y' + y = e^x$, $y(0) = 0$, by picard's method

Solution: By picard's method.

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$
$$= 0 + \int_0^x (e^x - y)\, dx$$

Here $x_0 = 0$, $y_0 = 0$

$$y^{(1)} = \int_0^x (e^x - 0)\, dx = e^x - 1$$

$$y^{(2)} = \int_0^x (e^x - e^x + 1)\, dx = x$$

$$y^{(3)} = \int_0^x (e^x - x)\, dx = e^x - x^2/2 - 1$$

$$y^{(4)} = \int_0^x \left[e^x - (e^x - x^2/2 - 1)\right] dx$$

$$= \frac{x^3}{6} + x$$

$$y^{(5)} = \int_0^x \left(e^x - x - \frac{x^3}{6}\right) dx$$

$$= e^x - x^2/2 - x^4/24 - 1$$

Approximate $y = e^x - x^2/2 - x^4/24 - 1$

11.9 Euler's method

To solve $\frac{dy}{dx} = f(x,y)$ put initial condition $y(x_0) = y_0$, Let $x = x_0, x_1, x_2, \dots$ where $h = x_i - x_{i-1}$

$$y_{n+1} = y_n + h f(x_n, y_n). \quad n = 0,1,2,\dots$$

This formula is called Euler's algorithm.

Ex: Using Euler's method solve numerically the equation

$y' = x+y,\ y(0) = 1$, for $x = 0.0\,(0.2)\,(1.0)$

solution:

Here $h = 0.2$, $f(x,y) = x+y$, $x_0 = 0$, $y_0 = 1$

$x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,\ x_5 = 1.0$

By Euler algorithm,

$$y_1 = y_0 + h f(x_0, y_0) = y_0 + h[x_0 + y_0]$$

$$= 1 + 0.2(0+1) = 1.2$$

$$\boxed{y_1 = 1.2}$$

$$y_2 = y_1 + h[x_1 + y_1] = 1.2 + 0.2(0.2+1.2) = 1.48$$

$$\boxed{y_2 = 1.48}$$

$$y_3 = y_2 + h[x_2 + y_2] = 1.48 + 0.2(0.4+1.48) = 1.85$$

$$\boxed{y_3 = 1.856}$$

$$y_4 = 1.856 + 0.2(0.6 + 1.856) = 2.3472$$

$$\boxed{y_4 = 2.3472}$$

$$y_5 = 2.3472 + 0.2(0.8 + 2.3472) = 2.94664$$

$$\boxed{y_5 = 2.94664}$$

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11.11. Modified Euler method

$$y_{n+1} = y_n + h\left[f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h\, f(x_n, y_n)\right)\right]$$

(or)

$$y(x+h) = y(x) + h\left[f\left(x + \tfrac{1}{2}h,\ y + \tfrac{1}{2}h\, f(x, y)\right)\right]$$

This formula is called Modified Euler's formula

Ex: compute y and $x = 0.25$ by modified Euler's method given $y' = 2xy$, $y(0) = 1$.

Solution:

$$f(x, y) = y' = 2xy$$

$$x_0 = 0, \quad y_0 = 1$$

$$x_1 = 0.25, \quad y_1 = ?$$

Take $h = 0.25$

$$y_{n+1} = y_n + h\left[f\left(x_n + \tfrac{h}{2},\ y_n + \tfrac{h}{2} f(x_n, y_n)\right)\right]$$

$$y_1 = 1 + (0.25)\left[f\left(0 + \tfrac{0.25}{2}\right),\ 1 + \tfrac{0.25}{2}\, 2(0)(1)\right]$$

$$= 1 + 0.25\left[f(0.125, 1)\right]$$

$$= 1 + 0.25\left[2(0.125)(1)\right]$$

$$= 1 + 0.0625$$

$$\boxed{y_1 = 1.0625}$$

11.12 Runge-Kutta method

second order for R.K method solve $\frac{dy}{dx} = f(x,$ given $y(x_0) = y_0$

$$k_1 = h\, f(x, y)$$

$$k_2 = h f\left[x + \tfrac{1}{2}h,\ y + \tfrac{1}{2}k_1\right]$$

and $\Delta y = k_2$, where $h = \Delta x$

Fourth order formula for R-K method

$$k_1 = h f(x, y)$$

$$k_2 = h f\left[x + \tfrac{1}{2}h,\ y + \tfrac{1}{2}k_1\right]$$

$$k_3 = h f\left[x + \tfrac{1}{2}h,\ y + \tfrac{1}{2}k_2\right]$$

$$k_4 = h f\left(x + h,\ y + k_3\right]$$

and $\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$$y(x+h) = y(x) + \Delta y$$

Ex: compute $y(0.3)$ given $\frac{dy}{dx} + y + xy^2 = 0$, $y(0) = 1$ by taking $h = 0.1$ by Runge-kutta method.

Solution: Given $\frac{dy}{dx} + y + xy^2 = 0$

$$\frac{dy}{dx} = -y - xy^2$$

$x_0 = 0,\ y_0 = 1$ $\qquad h = 0.1$

$x_1 = 0.3,\ y_1 = ?$

Second order formula

$$k_1 = h f(x_0, y_0)$$

$$k_1 = (0.1)\, f(0, 1)$$

$$= (0.1)\left(-1 - 0(1)^2\right)$$

$$= (0.1)(-1)$$

$$\boxed{k_1 = -0.1}$$

$$k_2 = h f\left(x_0 + h/2,\ y_0 + \frac{k_1}{2}\right)$$

$$= (0.1)\, f\left(0 + \frac{0.1}{2},\ 1 + \frac{(-0.1)}{2}\right)$$

$$= (0.1)\, f(0.05, 0.95)$$

$$= (0.1)\left[-0.95 - (0.05)(0.95)^2\right]$$

$$= (0.1)\left[-0.9951\right]$$

$$\boxed{k_2 = -0.0995}$$

$$\Delta y = k_2$$

$$\Delta y = y_1 - y_0 \Rightarrow y_1 = \Delta y + y_0$$

$$y_1 = -0.0995 + 1 \Rightarrow \boxed{y_1 = 0.9005}$$

Fourth order formula

$k_1 = hf(x_0, y_0)$

$k_1 = (0.1) f(0,1)$

$= (0.1)(-1-0(1)^2)$

$\boxed{k_1 = -0.1}$

$k_2 = hf\left(x_0 + h/2,\ y_0 + \frac{k_1}{2}\right)$

$= (0.1) f\left(0 + \frac{0.1}{2},\ 1 + \frac{(-0.1)}{2}\right)$

$= (0.1) f(0.05, 0.95)$

$= (0.1)\left[-0.95 - (0.05)(0.95)^2\right]$

$\boxed{k_2 = -0.0995}$

$k_3 = hf\left(x_0 + h/2,\ y_0 + \frac{k_2}{2}\right)$

$= 0.1 f\left[0 + \frac{0.1}{2},\ 1 + \frac{(-0.0995)}{2}\right]$

$= 0.1\ f(0.05, 0.9503)$

$= 0.1\left(-0.9503 - (0.05)(0.9503)^2\right]$

$\boxed{k_3 = -0.0996}$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1 f(0 + 0.1, 1 - 0.0996)$

$= (0.1) f(0.1, 0.9004)$

$= (0.1)\left[-0.9004 - (0.1)(0.9004)^2\right]$

$\boxed{k_4 = -0.0982}$

$\Delta y = 1/6\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= 1/6\left[-0.1 + 2(-0.0995) + 2(-0.0996) - 0.0982\right]$

$= 1/6\left[-0.1 - 0.1990 - 0.1992 - 0.0982\right]$

$\Delta y = -0.0994$

$\Delta y = y_1 - y_0$

$y_1 = \Delta y + y_0$

$y_1 = -0.0994 + 1$

$\boxed{y_1 = 0.9006}$

Again taking (x_1, y_1) in place of (x_0, y_0) repeat the process $k_1 = -0.0982$, $k_2 = -0.0960$, $k_3 = -0.962$,

$k_4 = -0.0934$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.9006 + \frac{1}{6}[-0.0982 + 2\times(-0.0960) + 2(-0.0962) + (-0.0924)]$

$\boxed{y(0.2) = 0.8046}$

Again, starting from (x_2, y_2) in place of (x_0, y_0)

$k_1 = -0.0934$, $k_2 = -0.0902$,

$k_3 = -0.0904$, $k_4 = -0.0867$

$\therefore y_3 = y_2 + \frac{1}{6}\Delta y = y_2 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$\boxed{y(0.3) = 0.7144.}$

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UNIT-V

Numerical solution of partial Differential Equations

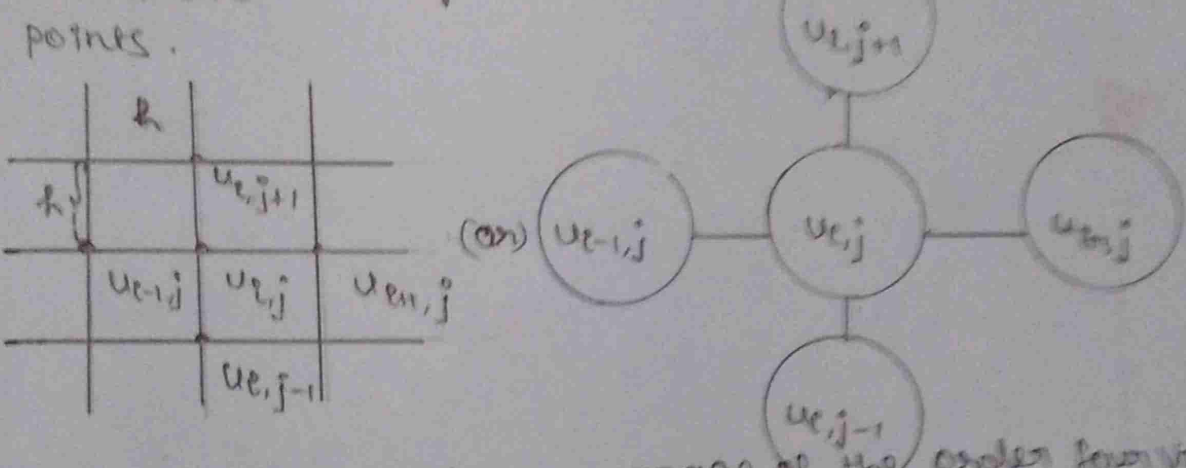
12.5 Elliptic equations

Laplace equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ that is $\nabla^2 u = 0$ or $u_{xx} + u_{yy} = 0$.

Standard five point formula

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right] \rightarrow (1)$$

That is, the value of u at any interior points is the arithmetic mean of the values of u at the four lattice points.



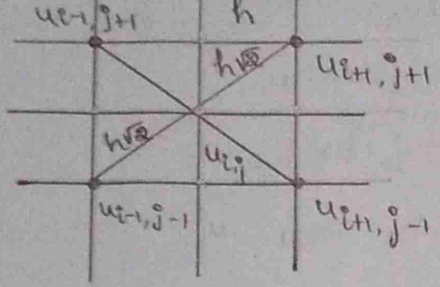
central value = average of the other four values

Diagonal five-point formula

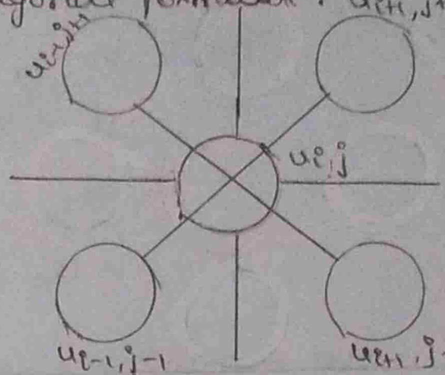
Instead of the formula (1) we can also use the formula

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$

which is called the diagonal five-point formula since this formula involves the values on the diagonals through $u_{i,j}$. since the Laplace equation is invariant in any coordinate system, the formula remains same when the coordinate axes are rotated through 45°. But the error in the diagonal formula is four times the error in the standard formula. Therefore, we always prefer the standard formula to the diagonal formula.



(or)

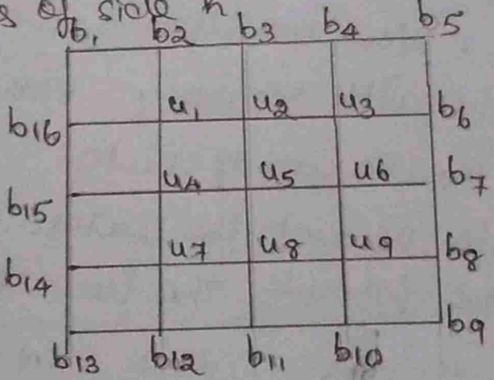


12.6 Solution of Laplace Equation: (By Liebmann's iteration process)

AIM

To solve the Laplace equation $u_{xx}+u_{yy}=0$ is in the bounded square region R with a boundary c when the boundary values of u are given on the boundary.

Let us divide the square region into a network of sub-squares of side h



The boundary values of u at the grid points are given and noted by $b_1, b_2, \ldots b_{16}$. The values of u at the interior lattice or grid points are assumed to be $u_1, u_2, \ldots u_9$

The start the iteration process, initially we find rough values at interior points and then we improve them by iterative process mostly using standard five point formula.

Find u_5 first : $u_5 = \frac{1}{4}(b_3+b_7+b_{11}+b_{15})$ (by SFPF)

Knowing u_5, we find u_1, u_3, u_7, u_9, that is the value at the centres of the four larger inner squares by using diagonal five point formula - DFPF

That is $u_1 = \frac{1}{4}(b_3+b_{15}+b_1+u_5)$

$$u_3 = \frac{1}{4}(b_5+u_5+b_3+b_7)$$

$$u_7 = \frac{1}{4}(u_5+b_{13}+b_{11}+b_{15})$$

$$u_9 = \frac{1}{4}(b_7+b_{11}+b_9+u_5)$$

The remaining 4 values u_2, u_4, u_6, u_8 can be got by using SFPF

That is, $u_2 = \frac{1}{4}(b_3+u_5+u_1+u_3)$

$$u_4 = \frac{1}{4}(u_1+u_7+u_5+b_{15})$$

$$u_6 = \frac{1}{4}(u_3+u_9+u_5+b_7)$$

$$u_8 = \frac{1}{4}(u_5+b_{11}+u_7+u_9)$$

Now we know all the boundary values of u and rough values of u at every grid point in the interior of the region R. Now we iterate the process and impro the values of u with accuracy. Start with u_5 and proceed to get the values of $u_1, u_2, \ldots u_9$ always using SFPF. taking into account the latest available value of u to use in the formula. The iterative formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right] \cdots \to$$

where the superscript of u denotes the iteration nu

Equation I is called Liebmann's iteration process

The process is stopped once we get the values with desired accuracy.

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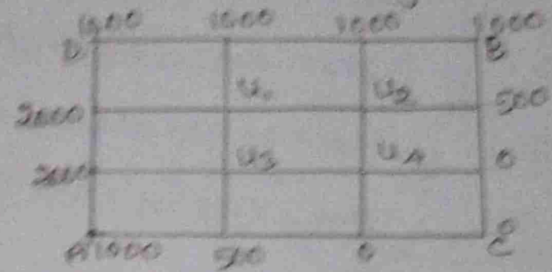
Ex: Evaluate the function $u(x,y)$ satisfying $\nabla^2 u = 0$ at the lattice points given the boundary values as follows.



Solution: Instead getting 4 equations in u_1, u_2, u_3 and u_4 and solving them for us we can assume some value u_4 and proceed iterative procedure; we can take $u_4 = 0$ and proceed or we take a value of $u_4 = 400$

Rough Values:

$$u_1 = (1000+2000+1000+400)/4 = 1100 \text{ (DFPF)}$$

$$u_2 = \frac{1}{4}(u_1 + u_4 + 1500) = 750 \quad \text{(SFPF)}$$

$$u_3 = \frac{1}{4}(u_1 + u_4 + 2500) = 1000 \quad \text{(SFPF)}$$

$$u_4 = \frac{1}{4}(u_2 + u_3) = 437.5 \quad \text{(SFPF)}$$

First Iteration: Here after we adopt only SFPF

$$u_1^{(1)} = \frac{1}{4}(750 + 1000 + 2000) = 1187.5$$

$$u_2^{(1)} = \frac{1}{4}(1187.5 + 437.5 + 1500) = 781.25$$

$$u_3^{(1)} = \frac{1}{4}(1187.5 + 437.5 + 2500) = 1031.25$$

$$u_4^{(1)} = \frac{1}{4}(781.25 + 1031.25) = 453.125$$

Second iteration:

$$u_1^{(2)} = \frac{1}{4}(781.25 + 1031.25 + 3000) = 1203.125$$

$$u_2^{(2)} = \frac{1}{4}(1203.125 + 453.125 + 1500) = 789.1$$

$$u_3^{(2)} = \frac{1}{4}(1203.125 + 453.125 + 2500) = 1039.1$$

$$u_4^{(2)} = \frac{1}{4}(789.1 + 1039.1) = 457.1$$

Third Iteration:

$$u_1^{(3)} = \frac{1}{4}(789.1 + 1039.1 + 3000) = 1207.1$$

$$u_2^{(3)} = \frac{1}{4}(1207.1 + 457.1 + 1500) = 791.1$$

$$u_3^{(3)} = \frac{1}{4}(1207.1 + 457.1 + 2500) = 1041.1$$

$$u_4^{(3)} = \frac{1}{4}(791.1 + 1041.1) = 458.1$$

Fourth iteration

$$u_1^{(4)} = 1/4(791.1 + 7041.1 + 3000) = 1208.1$$

$$u_2^{(4)} = 1/4(1208.1 + 458.1 + 1500) = 791.6$$

$$u_3^{(4)} = 1/4(1208.1 + 458.1 + 2500) = 1041.6$$

$$u_4^{(4)} = 1/4(791.6 + 1041.6) = 458.3$$

Fifth iteration:

$$u_1^{(5)} = \frac{1}{4}(791.6 + 1041.6 + 3000) = 1208.3$$

$$u_2^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 1500) = 791.7$$

$$u_3^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 2500) = 1041.7$$

$$u_4^{(5)} = \frac{1}{4}(791.7 + 1041.7) = 458.4$$

We are getting result correct to one decimal places. Further the increase in the value is < 0.1

We stop here. One mor iteration will give you the decision to make

$$\boxed{\therefore u_1 = 1208.3,\ u_2 = 791.7,\ u_3 = 1041.7,\ u_4 = 458.4}$$

12.8 Bender-Schmidt method

The one-dimensional heat equation, namely $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$ is an example of parabolic equation setting $\alpha^2 = \frac{1}{a}$, the equation becomes $\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0$

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Here $A = 1, B = 0, C = 0$ $\therefore B^2 - 4AC = 0$.
$\therefore$ It is parabolic at all points.

AIM: Our aim is to solve this by the method of finite differences.

To solve $u_{xx} = a u_t$ $\quad \cdots\cdots \to (1)$

with boundary condition

$$u(0,t) = T_0 \quad \cdots\cdots \to (2)$$

$$u(l,t) = T_1 \quad \cdots\cdots \to (3)$$

and with initial condition

$$u(x,0) = f(x), \quad 0 < x < l \quad \cdots\cdots \to (4)$$

We select a spacing h for the variable x and a spacing k for the time variable t

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

and $$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

Hence (1) becomes,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{a}{k}(u_{i,j+1} - u_{i,j})$$

$$\therefore u_{i,j+1} - u_{i,j} = \frac{k}{ah^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$= \lambda(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

where $\lambda = \frac{k}{ah^2}$

ie, $$\boxed{u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}} \quad \cdots\cdots \to (5)$$

Writing the boundary condition as $u_{0,j} = T_0$ $\cdots$
$u_{n,j} = T_1$ $\cdots$

where $nh = l$

and initial condition as

$$u_{i,0} = f(ih), \quad i = 1, 2, \ldots \quad \cdots\cdots \to (8)$$

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u is known at t = 0

Equation (5) facilitates to get the value of u at x_i and time t_{j+1}

Equation (5) is called explicit formula. It is valid if $0 < \lambda \le 1/2$

If we take $\lambda = 1/2$, the coefficient of $u_{i,j}$ vanishes hence equation (5) becomes,

$$\boxed{u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]}$$

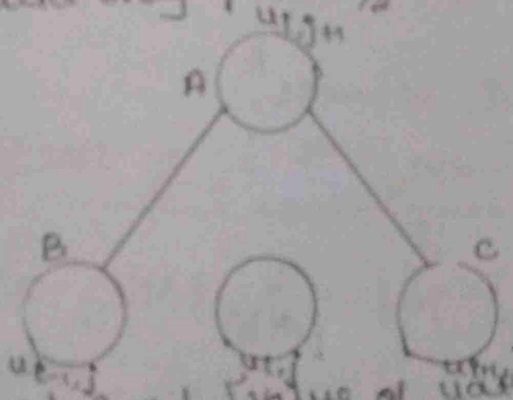
when

$$\lambda = \frac{1}{2} = \frac{k}{ah^2}, \quad \text{ie, } k = \frac{a}{2}h^2$$

ie, the value of u at $x = x_i$ at $t = t_{j+1}$ is equal to the average of the values of u the surrounding points x_{i-1} and x_{i+1} at the previous time t_j.

Equation (5) is called Bender-Schmidt recurrence equ

This is valid only if $k = \frac{a}{2}h^2$ (so select k like this)



Value of u at A = $\frac{1}{2}$ [Value of u at B + Value of u at C]

Ex: Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given

$u(0,t) = 0$, $u(4,t) = 0$, $u(x,0) = x(4-x)$. Assume $h = 1$.

Find the values of u upto $t = 5$

Solution

$u_{xx} = a u_t$, $\therefore a = 2$

To use Bender-Schmidt's equation, $k = \frac{a}{2}h^2 = 1$

Step-size in time $= k = 1$. The values of $u_{i,j}$ are tabulated below.

x-direction ⟶

t ↓ \ x →	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

t-direction ↓

Analysis: Range for $x : (0,4)$; for $t : (0,5)$

$u(x,0) = x(4-x)$. This gives $u(0,0) = 0$, $u(1,0) = 3$

$u(2,0) = 4$, $u(3,0) = 3$, $u(4,0) = 0$

For all t, at $x = 0$, $u = 0$ and for all t $x = 4$, $u = 0$

Using these values we fill up column under $x = 0, x$ and row against $t = 0$

$$\begin{array}{ccc} a & & b \\ & \searrow \swarrow & \\ & c & \end{array}$$

This means $c = \frac{a+b}{2}$

The values of u at $t = 1$ are written by seeing the val of u at $t = 0$ and using the average formula.

12.9 Crank – Nicholson Difference Method

AIM: To solve the parabolic equation

$u_{xx} = a u_t$ with boundary conditions

$u(0,t) = T_0$, $u(l,t) = T_1$ and the initial condi

$u(x,0) = f(x)$. The equation to be solved is $u_{xx} = a u_t$

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At $u_{i,j}$

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

and at $u_{i,j+1}$

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

Taking the average of these two values.

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1}}{2h^2}$$

using $u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$ equation ① reduces to

$$\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2} = a\frac{u_{i,j+1} - u_{i,j}}{k}$$

setting $\frac{k}{ah^2} = \lambda$, the above equation reduces to

$$\frac{1}{2}\lambda u_{i+1,j+1} + \frac{1}{2}\lambda u_{i-1,j+1} - (\lambda+1) u_{i,j+1} + (\lambda-1) u_{i,j}$$
$$= -\frac{1}{2}\lambda u_{i+1,j} - \frac{1}{2}\lambda u_{i-1,j} \quad ;\ldots\ldots(I)$$

Equation I is called Crank-Nicholson difference scheme or method.

Ex: solve by Crank-Nicholson method the equatio $u_{xx} = u_t$ subject to $u(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = t$ for two time steps.

Solution: x ranges from 0 to 1. Take $h = 1/4$; here a

$\therefore k = ah^2$ to use simple form

$$k = 1\left(\tfrac{1}{4}\right)^2 = \frac{1}{16}.$$

We use $u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}\right.$ $;\ldots\ldots$

(1)

	$x \to$ direction				
	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$t \downarrow$ $\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$
$\frac{3}{16}$	0				$\frac{3}{16}$

Let the unknowns be represented by $u_1, u_2, u_3, \ldots$

The boundary conditions are marked in the table against $t=0$, $x=0$ and $x=1$

Using the scheme (1)

$$u_1 = \tfrac{1}{4}(0+0+0+u_2) \qquad \text{i.e., } u_1 = \tfrac{1}{4}u_2 \longrightarrow (2)$$

$$u_2 = \tfrac{1}{4}(0+0+u_1+u_3) \qquad u_2 = \tfrac{1}{4}(u_1+u_3) \longrightarrow (3)$$

$$u_3 = \tfrac{1}{4}(0+0+u_1+\tfrac{1}{16}) \qquad u_3 = \tfrac{1}{4}(u_2+\tfrac{1}{16}) \longrightarrow (4)$$

Solving the three equations given by (2), (3), (4) we get u_1, u_2, u_3. Substitute u_3, u_1 values in (3).

$$u_2 = \tfrac{1}{4}\left[\tfrac{1}{4}u_2 + \tfrac{1}{4}(u_2+\tfrac{1}{16})\right]$$

$$u_2 = \frac{1}{224}\ 0.0045, \quad u_1 = \frac{1}{896} = 0.0011,$$

$$u_3 = 0.0168.$$

Similarly u_4, u_5, u_6 can be got again getting 3 equations in 3 unknown u_4, u_5, u_6.

We get $u_4 = 0.005899$, $u_5 = 0.01913$

$$u_6 = 0.05277$$

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